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Nambu–Poisson manifolds and associated n-ary Lie algebroids

José A Vallejo

Departament de Geometria i Topologia, Universitat de València, 46100 Burjassot (Valencia), Spain

E-mail: Jose.A.Vallejo@uv.es

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Abstract

We introduce an n-ary Lie algebroid canonically associated with a Nambu–Poisson manifold. We also prove that every Nambu–Poisson bracket defined on functions is induced by some differential operator on the exterior algebra, and characterizes such operators. Some physical examples are presented.

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1. Introduction

In the 1970s, trying to describe the simultaneous classical dynamics of three particles as a previous step towards a quantum statistics for the quark model, Nambu introduced a generalization of Poisson brackets formalism that today bears his name (see [Nam 73]). The field underwent a revitalization with the work of Takhtajan in the 1990s [Tak 94], which showed the algebraic setting underlying Nambu's ideas and introduced an analogue of the Jacobi identity, the fundamental identity, allowing the connection with the theory of n-ary Lie algebras developed by Filippov and others (see [Fil 85] and [Han-Wac 95]). In [AIP 97], further references relating to the development of this subject can be found, along with an alternative generalization of Poisson structures.

Recently, much work has been done in this area, showing an interesting algebraic structure (see, for instance, [Ale-Guh 96, Gau 96, Gau 98, Mic-Vin 96, MVV 98, Nak 98, Vin-Vin 98] and references therein). This is very interesting from both a mathematical and a physical point of view: an algebraic formulation not only provides a more concise and elegant framework, but also can be the source for new insights. In this direction, the notion of Leibniz algebroid was introduced [Dal-Tak 97, ILMP 99] as a kind of analogue of the Lie algebroid associated with a Poisson manifold. That concept of a Lie algebroid, has proved to be very useful in the study of Poisson manifolds and dynamics (for a sample, see [Lib 96, Cou 94, Wei 98, Gra-Urb 97]), and recently has been considered for the quantization of Poisson algebras [Lan-Ram 00]. The same is to be expected in the Nambu–Poisson case (n-ary Poisson algebras), with the appropriate generalizations.

Thus, the main motivation for this paper comes from the question: to what extent the constructions and properties of the Lie algebroids associated with Poisson manifolds, including quantization, carry over to the n-ary case? This paper deals with a very first step towards the answer, namely, the study of the canonical relation between Lie algebroids and Poisson manifolds in the n-ary case. For this purpose, one could consider Leibniz algebroids; however, this concept (although very interesting in itself) is not a genuine generalization of that of a Lie algebroid: it relies upon an algebraic construction called Leibniz (or Loday) algebra (see [Lod 93]), which is a non-commutative version of a Lie algebra. In [Gra-Mar 00] the notion of an n-ary Lie algebroid (called a Filippov algebroid by the authors) has been introduced, and it seems to better fit the aforementioned question.

In this paper, we construct an n-ary Lie algebroid canonically associated with a Nambu–Poisson manifold à la Koszul (see [Kos 85]), that is to say, by using differential operators on the (graded) exterior algebra of differential forms. An interesting advantage of this approach, is that it allows for the possibility of characterizing the operators which generate the brackets under consideration, this characterization being in terms of the commutator of the operator with the exterior differential. For the case of Poisson brackets and its extension as a graded brackets to the whole exterior algebra, such a study was made in [BMS 97], where these operators were called Jacobi operators (adopting the terminology from [Gra 92]), and essentially the same techniques will be used here to show that every Nambu–Poisson bracket coincides with the bracket induced on functions by a certain differential operator on the exterior algebra, giving its explicit form in terms of the Nambu–Poisson multivector. In the last section, some physical examples are described.

2. Basic definitions and results

Our main tool in the study of the n-ary generalizations of Poisson manifolds and Lie algebroids, will be the theory of differential operators on the exterior algebra of a manifold, so we collect here the basics of this theory.

Let $\mathrm{Hom}_{\mathbb{R}}(\Omega(M))$ be the space of $\mathbb{R}$-homomorphisms of the exterior algebra $F : \Omega(M) \longrightarrow \Omega(M)$. We say that F has $\mathbb{Z}$-degree $|F|$ (sometimes with F as an exponent) if F maps p-forms on $(p+|F|)$-forms, then we write $F \in \mathrm{Hom}_{\mathbb{R}}^{|F|}(\Omega(M))$ (or $F \in \mathrm{Hom}_{\mathbb{R}}^{F}(\Omega(M))$).

On the space $\mathrm{Hom}_{\mathbb{R}}(\Omega(M))$ we introduce a bracket $[.,.]$ (called a commutator) by means of

$$[F, G] = F \circ G - (-1)^{FG} G \circ F$$

and it is easy to prove that this bracket turns $(\mathrm{Hom}_{\mathbb{R}}(\Omega(M)), [.,.])$ into a graded Lie algebra.

A differential operator on $\Omega(M)$ of degree p and order q, is a homomorphism $D \in \mathrm{Hom}_{\mathbb{R}}^{p}(\Omega(M))$ such that

$$[[\ldots[[D, \mu_{a_0}], \mu_{a_1}], \ldots], \mu_{a_q}] = 0$$

for all $\mu_{a_i}, i \in \{0, \ldots, q\}$, where $a_i \in \Omega(M)$ and μ_{a_i} denotes the homomorphism multiplication by a_i, $\mu_{a_i}(b) = a_i \wedge b$, which has order 0 and degree $|a_i|$, and so will often be simply denoted a_i. The space of such operators is denoted $\mathcal{D}_q^p$; examples are the insertion operator $i_P \in \mathcal{D}_p^{-p}$ where $P \in \Gamma(\Lambda^p TM)$ is a p-multivector, and the generalized Lie derivative $\mathcal{L}_P = [i_P, d] \in \mathcal{D}_p^{-(p-1)}$, d being the exterior differential.

A useful result states that

$$[\mathcal{D}_q^p, \mathcal{D}_{q'}^{p'}] \subset \mathcal{D}_{q+q'-1}^{p+p'}. \tag{1}$$

Also, we have that any operator $D \in \mathcal{D}_q^{-q}$ has the form i_A, for some $A \in \Gamma(\Lambda^q TM)$. This is a consequence of (1) and the fact that any operator is determined by its action on forms with degree equal or less than the order of the operator.

The order defines a filtration on the space of all differential operators. Indeed, given $p, k \in \mathbb{Z}$ with $p > k$, if D is an operator of order $\leqslant k$, then it is also of order $\leqslant p$, that is to say

$$\{0\} = \mathcal{D}_{-1}(M) \subset \mathcal{D}_0(M) \subset \cdots \subset \mathcal{D}_k(M) \subset \mathcal{D}_{k+1}(M) \subset \cdots.$$

Definition 1. *Given $k \in \mathbb{Z}$, we will call the symbol of order k of M the $\Omega(M)$-module*

$$\mathrm{Symb}_k(M) = \frac{\mathcal{D}_k(M)}{\mathcal{D}_{k-1}(M)}$$

and the space of symbols of the exterior algebra $\Omega(M)$ is defined as the graded $\Omega(M)$-module

$$\mathrm{Symb}(M) = \bigcup_{k\in\mathbb{Z}} \mathrm{Symb}_k(M).$$

Note that, if D is a differential operator of order $\leqslant k$, $D \in \mathcal{D}_k(M)$, then we can speak about the symbol of order i of the operator D, that is to say,

$$\mathrm{Symb}_i(D) \in \frac{\mathcal{D}_k(M)}{\mathcal{D}_{k-1}(M)} \qquad i \leqslant k.$$

We will need some terminology from the theory of Nambu–Poisson manifolds.

A Nambu–Poisson bracket on a manifold M, is an n-linear mapping $\{.,\ldots,.\}$: $C^\infty(M) \times \cdots \times C^\infty(M) \longrightarrow C^\infty(M)$, satisfying:

(1) Skew-symmetry. For all $f_i \in C^\infty(M)$, $(1 \leqslant i \leqslant n)$ and $\sigma \in S_n$ (S_n is the symmetric group of order n)

$$\{f_1, \ldots, f_n\} = (-1)^{\varepsilon(\sigma)}\{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}.$$

(2) The Leibniz rule. For all $f_i, g \in C^\infty(M)$, $(1 \leqslant i \leqslant n)$

$$\{f_1 g, f_2, \ldots, f_n\} = f_1\{g, f_2, \ldots, f_n\} + g\{f_1, f_2, \ldots, f_n\}.$$

(3) The fundamental identity. For all $f_i, g_j \in C^\infty(M)$, $(1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n)$

$$\{f_1, f_2, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{j=1}^{n}\{g_1, \ldots, \{f_1, \ldots, f_{n-1}, g_j\}, \ldots, g_n\}.$$

If $\{.,\ldots,.\}$ is a Nambu–Poisson bracket, it has an associated n-multivector $P \in \Gamma(\Lambda^n TM)$ through

$$P(df_1 \wedge \cdots \wedge df_n) = \{f_1, \ldots, f_n\}$$

which is called the Nambu–Poisson multivector.

A Nambu–Poisson manifold is a pair $(M, \{.,\ldots,.\})$ where $\{.,\ldots,.\}$ is a Nambu–Poisson bracket (which also can be denoted (M, P)).

A Lie (or Filippov) n-algebra structure on a vector space V, is an n-linear skew-symmetric bracket $[.,\ldots,.]$ satisfying the generalized Jacobi identity, also called the fundamental identity:

$$[v_1, v_2, \ldots, v_{n-1}, [u_1, \ldots, u_n]] = \sum_{j=1}^{n}[u_1, \ldots, [v_1, \ldots, v_{n-1}, u_j], \ldots, u_n]$$

for all $v_i, u_j \in V$, $(1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n)$.

A Lie (or Filippov) n-algebroid is a vector bundle $p : E \longrightarrow M$ equipped with an n-bracket $[.,\ldots,.]$ on sections of E ($V = \Gamma(E)$) and a vector bundle morphism $q : \Lambda^{n-1}E \longrightarrow TM$ over the identity on M, called the anchor map, such that:

(1) The induced morphism on sections $q : \Gamma(\Lambda^{n-1}E) \longrightarrow \Gamma(TM)$ satisfies the following relation with respect to the bracket of vector fields:

$$[q(X_1 \wedge \cdots \wedge X_{n-1}), q(Y_1 \wedge \cdots \wedge Y_{n-1})] = \sum_{i=1}^{n-1} q(Y_1, \cdots [X_1, \cdots, X_{n-1}, Y_i] \wedge \cdots \wedge Y_{n-1}).$$

and

(2) $$[X_1, \ldots, X_{n-1}, fY] = f[X_1, \ldots, X_{n-1}, Y] + q(X_1 \wedge \cdots \wedge X_{n-1})(f)Y$$

for all $X_1, \ldots, X_{n-1}, Y \in \Gamma(E)$ and $f \in C^\infty(M)$.

We will call these structures n-Lie brackets and algebroids, respectively, for short.

3. n-Lie brackets on 1-forms induced by differential operators

In [Kos 85], Koszul introduces the following notation: for $D \in \mathcal{D}_n$ (of any degree)

$$\Phi^n_D(a_1, \ldots, a_n) = [[\ldots[D, a_1], \ldots], a_n](1). \tag{2}$$

In fact, he considers that expression for $D \in \mathcal{D}_2^{-1}$ such that $D(1) = 0$, and uses it in order to define a bracket on $\Omega(M)$ as

$$\begin{aligned}[\![a, b]\!]_D &= (-1)^a \Phi^2_D(a, b) = (-1)^a[[D, a], b](1) \\ &= (-1)^a D(ab) - (-1)^a D(a)b - (-1)^{a(D+1)} a D(b).\end{aligned} \tag{3}$$

Koszul also studies under what conditions the bracket $[\![., .]\!]_D$ is a graded Lie one, and obtains the necessary and sufficient condition

$$D^2 \in \mathcal{D}_2$$

when *a priori* D^2 lies in $\mathcal{D}_3$. This idea can be followed on to construct new kinds of brackets from a differential operator.

Remark 2. In the following, we will only consider operators such that $D(1) = 0$.

By a direct calculation involving only the Jacobi identity and the skew symmetry for the commutator $[., .]$, it can be shown that, for any $D \in \mathcal{D}^D_n$

$$\begin{aligned}&\Phi^n_D(a_1, \ldots, a_{n-1}, \Phi^n_D(b_1, \ldots, b_n)) \\ &\quad = (-1)^{(D+\sum_{i=1}^{n-1} a_i)D} \Phi^n_D(\Phi^n_D(a_1, \ldots, a_{n-1}, b_1), b_2, \ldots, b_n) \\ &\quad\quad +(-1)^{(D+\sum_{i=1}^{n-1} a_i)(D+b_1)} \Phi^n_D(b_1, \Phi^n_D(a_1, \ldots, a_{n-1}, b_2), b_3, \ldots, b_n) \\ &\quad\quad + \cdots + (-1)^{(D+\sum_{i=1}^{n-1} a_i)(D+\sum_{j=1}^{n-1} b_j)} \Phi^n_D(b_1, \ldots, b_{n-1}, \Phi^n_D(a_1, \ldots, a_{n-1}, b_n)) \\ &\quad\quad +[[\ldots[[[[\ldots[D, a_1], \ldots], \ldots], a_{n-1}], D], b_1], \ldots], b_n](1).\end{aligned} \tag{4}$$

If we want to induce a n-ary bracket on 1-forms, we need an operator of type $D \in \mathcal{D}^{-(n-1)}_n(\Omega(M))$ (so acting on n 1-forms gives a 1-form, recall (1)). In this case, with $a_i, b_j \in \Omega^1(M)$, (4) reduces to

$$\begin{aligned}&\Phi^n_D(a_1, \ldots, a_{n-1}, \Phi^n_D(b_1, \ldots, b_n)) \\ &\quad = (-1)^{(D+n-1)D} \Phi^n_D(\Phi^n_D(a_1, \ldots, a_{n-1}, b_1), b_2, \ldots, b_n) \\ &\quad\quad +(-1)^{(D+n-1)(D+1)} \Phi^n_D(b_1, \Phi^n_D(a_1, \ldots, a_{n-1}, b_2), b_3, \ldots, b_n) \\ &\quad\quad + \cdots + (-1)^{(D+n-1)(D+n-1)} \Phi^n_D(b_1, \ldots, b_{n-1}, \Phi^n_D(a_1, \ldots, a_{n-1}, b_n)) \\ &\quad\quad +[[\ldots[[[[\ldots[D, a_1], \ldots], \ldots], a_{n-1}], D], b_1], \ldots], b_n](1)\end{aligned}$$

and, as the operator degree is $|D| = -(n-1)$, to

$$\begin{aligned}\Phi^n_D(a_1, \ldots, a_{n-1}, \Phi^n_D(b_1, \ldots, b_n)) \\ &= \Phi^n_D(\Phi^n_D(a_1, \ldots, a_{n-1}, b_1), b_2, \ldots, b_n) \\ &\quad + \Phi^n_D(b_1, \Phi^n_D(a_1, \ldots, a_{n-1}, b_2), b_3, \ldots, b_n) \\ &\quad + \cdots + \Phi^n_D(b_1, \ldots, b_{n-1}, \Phi^n_D(a_1, \ldots, a_{n-1}, b_n)) \\ &\quad + [[\ldots[[[[\ldots[D, a_1], \ldots], \ldots], a_{n-1}], D], b_1], \ldots], b_n](1). \end{aligned} \tag{5}$$

Taking into account that (for odd degree operators) $[[D, a], D] = \frac{1}{2}[D^2, a]$, we see that this expression generalizes the one given by Koszul (in lemma 1.5 of [Kos 85]) for the Jacobi Identity when only 1-forms are considered:

$$\Phi^2_D(a_1, \Phi^2_D(a_2, a_3)) = \Phi^2_D(\Phi^2_D(a_1, a_2), a_3) + \Phi^2_D(a_2, \Phi^2_D(a_1, a_3)) + \tfrac{1}{2}\Phi^3_{D^2}(a_1, a_2, a_3).$$

Thus, we are led to consider the following definition for an n-ary bracket on 1-forms, induced by an operator $D \in \mathcal{D}_n^{-(n-1)}$.

Definition 3. *For $D \in \mathcal{D}_n^{-(n-1)}$ and $a_i \in \Omega^1(M)$, $i \in \{i, \ldots, n\}$,*

$$[\![a_1, \ldots, a_n]\!]_D = \Phi^n_D(a_1, \ldots, a_n) = [[\ldots[D, a_1], \ldots], a_n](1). \tag{6}$$

Proposition 4. *Let $[\![., \ldots, .]\!]_D$ the bracket on 1-forms induced by an operator $D \in \mathcal{D}_n^{-(n-1)}$. Then, it has the following properties:*

(i) $\mathbb{R}$-linearity on each argument;
(ii) skew-symmetry.

Proof.

(i) Is a direct consequence of the corresponding property for the bracket on $\mathrm{Hom}_{\mathbb{R}}(\Omega(M))$.
(ii) Here we use that, for any $F \in \mathrm{Hom}_{\mathbb{R}}(\Omega(M))$, and $a \in \Omega^{|a|}(M)$, $b \in \Omega^{|b|}(M)$,

$$[[F, a], b] = (-1)^{ab}[[F, b], a]$$

and the statement follows from a straightforward computation. □

In order to construct an n-Lie bracket on 1-forms, we need a fundamental identity. A glance at (5) tells us we are almost done, it suffices to give a condition on D similar to that of Koszul for the binary case.

Proposition 5. *Let $D \in \mathcal{D}_n^{-(n-1)}$ be a differential operator. Then, $[\![., \ldots, .]\!]_D$ induces an n-Lie algebra structure on $\Omega^1(M)$ if and only if, for any $a_1, \ldots, a_{n-1} \in \Omega^1(M)$,*

$$[[[\ldots[D, a_1], \ldots], a_{n-1}], D] \in \mathcal{D}_{n-1}.$$

Proof. Just observe that the condition above kills the last term in (5). □

4. n-Lie algebroids

In this section, we will construct an n-Lie algebroid on the space of 1-forms. Given an operator $D \in \mathcal{D}_n^{-(n-1)}$, we consider an extension of the bracket $[\![., \ldots, .]\!]_D : \Omega^1(M) \times \overset{n)}{\cdots} \times \Omega^1(M) \longrightarrow \Omega^1(M)$ to another one defined in $\Omega^1(M) \times \overset{n-1)}{\cdots} \times \Omega^1(M) \times \Omega(M)$, that is, where the last argument is a differential form of any degree, with the same formula:

$$[\![a_1, \ldots, a_{n-1}, b]\!]_D = \Phi_D^n(a_1, \ldots, a_{n-1}, b)$$

$\forall a_i \in \Omega^1(M), i \in \{1, \ldots, n-1\}, b \in \Omega(M)$. Although this bracket loses the skew-symmetry property, under the condition on D given in proposition 2 it retains the fundamental identity, in the sense that (recall (4))

$$\begin{aligned}\Phi_D^n(a_1, \ldots, a_{n-1}, \Phi_D^n(b_1, \ldots, b_{n-1}, f)) &\\ = \Phi_D^n(\Phi_D^n(a_1, \ldots, a_{n-1}, b_1), b_2, \ldots, b_{n-1}, f) &\\ + \cdots + \Phi_D^n(b_1, \ldots, b_{n-1}, \Phi_D^n(a_1, \ldots, a_{n-1}, f)) &\end{aligned}$$

when $f \in C^\infty(M), a_i, b_i \in \Omega^1(M), i \in \{1, \ldots, n-1\}$. This will enable us to construct an n-Lie (or Filippov) algebroid associated with D.

Indeed, the Leibniz property guarantees that $[\![a_1, \ldots, a_{n-1}, .]\!]_D$ acts as a derivation on $C^\infty(M)$, that is, it belongs to $\Gamma(TM)$: for any $f, g \in C^\infty(M), a_i \in \Omega^1(M), i \in \{1, \ldots, n-1\}$,

$$[\![a_1, \ldots, a_{n-1}, fg]\!]_D = f[\![a_1, \ldots, a_{n-1}, g]\!]_D + [\![a_1, \ldots, a_{n-1}, f]\!]_D g.$$

Now consider the cotangent bundle $T^*M \xrightarrow{\pi} M$. On the sections $\Omega^{n-1}(M) = \Gamma(\Lambda^{n-1}(T^*M))$, we define the anchor map as

$$\begin{aligned}&q : \Omega^{n-1}(M) \longrightarrow \Gamma(TM)\\ &a_1 \wedge \cdots \wedge a_{n-1} \mapsto q(a_1 \wedge \cdots \wedge a_{n-1}) = [\![a_1, \ldots, a_{n-1}, .]\!]_D\end{aligned}$$

and extend it by linearity. Let us check that the definition makes sense: the observation above tells us that for $f, g \in C^\infty(M)$,

$$q(a_1 \wedge \cdots \wedge a_{n-1})(fg) = fq(a_1 \wedge \cdots \wedge a_{n-1})(g) + gq(a_1 \wedge \cdots \wedge a_{n-1})(f)$$

and so q takes values in the right space. Note also the $C^\infty(M)$-linearity of the anchor map, obtained as a consequence of the Leibniz property for the bracket on $\mathrm{Hom}_{\mathbb{R}}(\Omega(M))$ and the degree of D: if $f, g \in C^\infty(M), a_i \in \Omega^1(M), i \in \{1, \ldots, n-1\}$,

$$\begin{aligned}q(a_1 \wedge \cdots \wedge fa_i \wedge \cdots \wedge a_{n-1})(g) &= [\![a_1, \ldots, fa_i, \ldots, a_{n-1}, g]\!]_D\\ &= [[\ldots[[\ldots[D, a_i], \ldots], fa_i], \ldots], g](1)\\ &= f[[\ldots[[\ldots[D, a_1], \ldots], a_i], \ldots], g](1) + (-1)^{a_i(n-i-1)}\\ &\quad \times[[\ldots[[\ldots[D, a_1], \ldots], f], \ldots], g](a_i)\\ &= fq(a_1 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_{n-1})(g)\end{aligned}$$

(the term with f, g inside the brackets vanishes because $|D| = -(n-1)$).

Let us now verify the conditions of the n-Lie algebroid definition. One of these is nothing more but the property (ii) of proposition 1:

$$\begin{aligned}[\![a_1, \ldots, a_{n-1}, fb]\!]_D &= f[\![a_1, \ldots, a_{n-1}, b]\!]_D + [\![a_1, \ldots, a_{n-1}, f]\!]_D b\\ &= f[\![a_1, \ldots, a_{n-1}, b]\!]_D + q(a_1 \wedge \cdots \wedge a_{n-1})(f)b\end{aligned}$$

and the other one is equivalent to the fundamental identity. On the one hand we have

$$\begin{aligned}[][q(a_1 \wedge \cdots \wedge a_{n-1}), q(b_1 \wedge \cdots \wedge b_{n-1})](f) &= q(a_1 \wedge \cdots \wedge a_{n-1})[\![b_1, \ldots, b_{n-1}, f]\!]_D \\ &\quad - q(b_1 \wedge \cdots \wedge b_{n-1})[\![a_1, \ldots, a_{n-1}, f]\!]_D \\ &= [\![a_1, \ldots, a_{n-1}, [\![b_1, \ldots, b_{n-1}, f]\!]_D]\!]_D \\ &\quad - [\![b_1, \ldots, b_{n-1}, [\![a_1, \ldots, a_{n-1}, f]\!]_D]\!]_D\end{aligned}$$

and on the other

$$\begin{aligned}&\sum_{i=1}^{n-1} q(b_1 \wedge \cdots \wedge [\![a_1, \ldots, a_{n-1}, b_i]\!]_D \wedge \cdots \wedge b_{n-1})(f) \\ &\qquad = \sum_{i=1}^{n-1} [\![b_1, \ldots, [\![a_1, \ldots, a_{n-1}, b_i]\!]_D, \ldots b_{n-1}, f]\!]_D.\end{aligned}$$

The fundamental identity equates these two expressions, so

$$\begin{aligned}&[q(a_1 \wedge \cdots \wedge a_{n-1}), q(b_1 \wedge \cdots \wedge b_{n-1})](f) \\ &\qquad = \sum_{i=1}^{n-1} q(b_1 \wedge \cdots \wedge [\![a_1, \ldots, a_{n-1}, b_i]\!]_D \wedge \cdots \wedge b_{n-1})(f).\end{aligned}$$

We summarize all this in the following result.

Theorem 6. *Let $D \in \mathcal{D}_n^{-(n-1)}$ be such that $[[[\ldots[D, a_1], \ldots], a_{n-1}], D] \in \mathcal{D}_{n-1}$, $\forall a_i \in \Omega^1(M)$, $i \in \{1, \ldots, n-1\}$. Then $\left(T^*M \xrightarrow{\pi} M, [\![., \ldots, .]\!]_D, q\right)$ is an n-Lie algebroid.*

Remark 7. Given an operator D as in the previous proposition, any isomorphism $L : T^*M \longrightarrow TM$ (for example, the canonical ones associated to Riemannian, Poisson or symplectic manifolds) induces the corresponding n-Lie algebroid on $\left(TM \xrightarrow{\pi} M, [\![., \ldots, .]\!]_{\tilde{D}}, q\right)$, where $\tilde{D} = \tilde{L} \circ D \circ \tilde{L}^{-1}$ and $\tilde{L}$ is the extension of L as a homomorphism of exterior algebras.

5. The canonical n-Lie algebroid associated with a Nambu–Poisson manifold

Our goal in this section is to construct a basic example of n-Lie algebroid, and we shall obtain a result similar to that for Poisson manifolds (namely, that each Poisson manifold has a canonical Lie algebroid structure associated with it) for the n-ary case, i.e each n-Poisson (Nambu–Poisson) manifold has a canonically associated n-Lie algebroid.

Let $(M, \{., \ldots, .\})$ be a Nambu–Poisson manifold. The Leibniz property for $\{., \ldots, .\}$ tells us that we can express it through a n-multivector $P \in \Gamma(\Lambda^n TM)$ (called the Nambu–Poisson multivector) such that, for any $f_1, \ldots, f_n \in C^\infty(M)$:

$$\{f_1, \ldots, f_n\} = P_{f_1 \ldots f_n} = i_{df_n \wedge \cdots \wedge df_1} P.$$

Let us translate the fundamental identity for $\{., \ldots, .\}$ in terms of P. Consider a Hamiltonian vector field, which in this case will have the form

$$P_{f_1 \ldots f_{n-1}} \in \mathcal{X}(M)$$

and let us compute the Lie derivative of $P \in \Gamma(\Lambda^n TM)$, $\mathcal{L}_{P_{f_1 \cdots f_{n-1}}} P$, which is a tensor of the same type of P and, accordingly, will act on n functions $g_1, \ldots, g_n \in C^\infty(M)$:

$$\left(\mathcal{L}_{P_{f_1 \cdots f_{n-1}}} P\right)(g_1, \ldots, g_n)$$

$$= \mathcal{L}_{P_{f_1 \cdots f_{n-1}}}(P_{g_1,\ldots,g_n}) - \sum_{i=1}^{n} P\left(g_1, \ldots, \mathcal{L}_{P_{f_1 \cdots f_{n-1}}} g_i, \ldots, g_n\right)$$

$$= \mathcal{L}_{P_{f_1 \cdots f_{n-1}}}(\{g_1, \ldots, g_n\}) - \sum_{i=1}^{n} \{g_1, \ldots, \mathcal{L}_{P_{f_1 \cdots f_{n-1}}} g_i, \ldots, g_n\}$$

$$= \{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} - \sum_{i=1}^{n} \{g_1, \ldots, \{f_1, \ldots, f_{n-1}, g_i\}, \ldots, g_n\}.$$

Thus, the fundamental identity for the n-bracket $\{.,\ldots,.\}$, translates into

$$\mathcal{L}_{P_{f_1 \cdots f_{n-1}}} P = 0 \qquad \forall f_i \in C^\infty(M) \qquad i \in \{1, \ldots, n-1\}.$$

Now, this Lie derivative is nothing but the Schouten–Nijenhuis bracket of $P_{f_1 \ldots f_{n-1}}$ and P (taken as multivectors), so we can write

$$[P_{f_1 \ldots f_{n-1}}, P]_{\mathrm{SN}} = 0.$$

Knowing what the fundamental identity means in terms of P, we can construct the promised example. In order to do this, consider the operator

$$D = \mathcal{L}_P \in \mathcal{D}_n^{-(n-1)}$$

which has the order and degree we have seen can generate Filippov brackets on 1-forms.

Remark 8. An easy computation (using that $d^2 = 0$ and $[i_{\text{multivector}}, f] = 0$) gives

$$[\mathcal{L}_P, f] = [[i_P, d], f] = [i_P, [d, f]] - (-1)^2[d, [i_P, f]] = [i_P, df] = i_{i_{df} P}$$

and similarly (using here that $\mathcal{L}_P$ and d commute)

$$[\mathcal{L}_P, df] = [\mathcal{L}_P, [d, f]] = -[d, [\mathcal{L}_P, f]] = -[d, i_{i_{df} P}] = \mathcal{L}_{i_{df} P}.$$

As a consequence we get the following result.

Theorem 9. *For each $P \in \Gamma(\Lambda^n TM)$ satisfying $[P_{f_1 \ldots f_n}, P]_{\mathrm{SN}} = 0$ (for all $f_i \in C^\infty(M)$, $i \in \{1, \ldots, n-1\}$), the operator $D = \mathcal{L}_P \in \mathcal{D}_n^{-(n-1)}$ induces an n-Lie bracket on 1-forms, and so each Nambu–Poisson bracket on functions has an associated n-Lie bracket on $\Omega^1(M)$.*

Proof. By Leibniz property, we only need to check that

$$[\mathcal{L}_P, [[\ldots[[\mathcal{L}_P, a_1], a_2], \ldots], a_{n-1}]] \in \mathcal{D}_{n-1}$$

when the $a_i' s$ are of the form $f \in C^\infty(M)$, $df \in \Omega^1(M)$ (in fact, we will see that this expression vanishes). Now, we can distinguish three different cases.

1st case: there are at least two functions among the a_i ($i \in \{1, \ldots, n-1\}$).

Rearrange the factors to get

$$[\mathcal{L}_P, [[\ldots[[[\mathcal{L}_P, f_1], f_2], df_3], \ldots], df_{n-1}]]$$

and then compute, having in mind the previous remark,

$$[\mathcal{L}_P, [[\ldots[[[\mathcal{L}_P, f_1], f_2], df_3], \ldots], df_{n-1}]] = [[\ldots[[i_{i_{df} P}, f_2], df_3], \ldots], df_{n-1}] = 0.$$

2nd case: there is exactly one function among the a_i ($i \in \{1, \ldots, n-1\}$).

This time, rearrange to

$$\begin{aligned}
&[\mathcal{L}_P, [[[\ldots[[\mathcal{L}_P, df_1], df_2], \ldots], df_{n-2}], f_{n-1}]] \\
&\quad = [\mathcal{L}_P, [[[\ldots[\mathcal{L}_{i_{df_1} P}, df_2], \ldots], df_{n-2}], f_{n-1}]] \\
&\quad = \cdots = [\mathcal{L}_P, [\mathcal{L}_{i_{df_{n-2} \wedge \cdots \wedge df_1} P}, f_{n-1}]] = [\mathcal{L}_P, [\mathcal{L}_{P_{f_1 \cdots f_{n-2}}}, f_{n-1}]] \\
&\quad = [\mathcal{L}_P, i_{i_{df_{n-1}} P_{f_1 \cdots f_{n-2}}}] = [\mathcal{L}_P, i_{P_{f_1 \cdots f_{n-1}}}] = i_{[P_{f_1 \cdots f_{n-1}}, P]_{\mathrm{SN}}} = 0.
\end{aligned}$$

3rd case: there is no function among the a_i ($i \in \{1, \dots, n-1\}$).
Here, all we have are exact 1-forms, and then

$$\begin{aligned}
&[\mathcal{L}_P, [[\dots[[\mathcal{L}_P, df_1], df_2], \dots], df_{n-1}]] \\
&\quad = [\mathcal{L}_P, [[\dots[\mathcal{L}_{i_{df_1}P}, df_2], \dots], df_{n-1}]] \\
&\quad = \cdots = [\mathcal{L}_P, \mathcal{L}_{i_{df_{n-1}\wedge\cdots\wedge df_1}P}] = [\mathcal{L}_P, \mathcal{L}_{P_{f_1\cdots f_{n-1}}}] = \mathcal{L}_{[P_{f_1\cdots f_{n-1}}, P]_{\mathrm{SN}}} = 0.
\end{aligned}$$

□

Corollary 10. *Let $(M, \{.,\dots,.\})$ be a Nambu–Poisson manifold and $P \in \Gamma(\Lambda^n TM)$ the induced Nambu–Poisson tensor. Then $\left(T^*M \xrightarrow{\pi} M, [\![.,\dots,.]\!]_{\mathcal{L}_P}, q\right)$ constructed as above is an n-Lie algebroid.*

6. Nambu–Poisson structures induced by differential operators

We have just seen how to construct Filippov algebroids from Nambu–Poisson structures. Next, we would like to obtain examples of the later ones also by using differential operator techniques.

Given an operator $D \in \mathcal{D}_n^{-(n-1)}$ and a function $f \in C^\infty(M)$, we define a bracket

$$\begin{aligned}
[\![.,\dots,.]\!]_{D_f} &: \Omega^1(M)\times \overset{n-1}{\cdots} \times\Omega^1(M) \longrightarrow C^\infty(M) \\
(a_1,\dots,a_{n-1}) &\longmapsto [\![a_1,\dots,a_{n-1}]\!]_{D_f} = \Phi_D^n(a_1,\dots,a_{n-1}, f)
\end{aligned}$$

and from it, an n-bracket on functions

$$\begin{aligned}
&\{.,\dots,.\}_D : C^\infty(M)\times \overset{n}{\cdots} \times C^\infty(M) \longrightarrow C^\infty(M) \\
&(f_1,\dots,f_n) \longmapsto \{f_1,\dots,f_n\}_D = [\![df_1,\dots,df_{n-1}]\!]_{D_{f_n}} = \Phi_D^n(df_1,\dots,df_{n-1}, f_n).
\end{aligned}$$

This bracket is linear in each argument and that it has the Leibniz property. The following result, specifies other features.

Proposition 11. *Let $D \in \mathcal{D}_n^{-(n-1)}$ and $\{.,\dots,.\}_D$ as above. Then:*

(i) $\{.,\dots,.\}_D$ *is skew-symmetric if and only if* $\mathrm{Symb}_n([D,d]) = 0$.
(ii) $\{.,\dots,.\}_D$ *verifies the fundamental identity if and only if for all $a_1,\dots,a_{n-1} \in \Omega^1(M)$, then $[D, [[\dots[D, a_1], \dots], a_{n-1}]] \in \mathcal{D}_{n-1}$.*

Proof. Condition (ii) is known from section 1 (see (5)). For the condition (i) to be understood, we only need to check it. First, note that if $i < n-1$, then clearly $\{f_1,\dots,f_i,f_{i+1},\dots,f_n\} = -\{f_1,\dots,f_{i+1},f_i,\dots,f_n\}$. Next, consider any differential operator Δ and $f, g \in C^\infty(M)$; we have

$$\begin{aligned}
[[\Delta, df], g] &= [[\Delta, [d, f]], g] \\
&= [[[\Delta, d], f], g] + (-1)^\Delta [d, [[\Delta, f], g]] - [[\Delta, f], dg]
\end{aligned}$$

thus, if we take $\Delta = [[\dots[D, df_1], \dots], df_{n-2}] \in \mathcal{D}_2^{-1}(\Omega(M))$, $f = f_{n-1}$, $g = f_n \in C^\infty(M)$ it results $[[\Delta, f], g] = 0$, and so

$$\begin{aligned}
[[\Delta, df_{n-1}], f_n] &= [[[\Delta, d], f_{n-1}], f_n] - [[\Delta, f_{n-1}], df_n] \\
&= [[[\Delta, d], f_{n-1}], f_n] - [[\Delta, df_n], f_{n-1}]. \qquad (7)
\end{aligned}$$

It is the first term on the last member which destroys skew symmetry, but writing

$$\Delta_{k-1} = [[\dots[D, df_1], \dots], df_{k-1}]$$

we see that

$$\begin{aligned}[\Delta_k, d] &= [[[\ldots[D, df_1], \ldots], df_k], d] = [[[\ldots[D, df_1], \ldots], [d, f_k]], d] \\ &= [[\Delta_{k-1}, [d, f_k]], d] = -(-1)^{\Delta_{k-1}}[df_k, [\Delta_{k-1}, d]] \\ &= [[\Delta_{k-1}, d], df_k] \end{aligned} \tag{8}$$

thus, we have from (7) and (8),

$$\begin{aligned}\{f_1, \ldots, f_{n-1}, f_n\} &= [[[\ldots[D, df_1], \ldots], df_{n-1}], f_n](1) \\ &= [[[\ldots[[D, d], df_1], \ldots], f_{n-1}], f_n](1) - [[[\ldots[D, df_1], \ldots], df_n], f_{n-1}](1) \\ &= [[[\ldots[[D, d], df_1], \ldots], df_{n-1}], f_n](1) - \{f_1, \ldots, f_n, f_{n-1}\}.\end{aligned}$$

□

Corollary 12. *Let $D \in \mathcal{D}_n^{-(n-1)}$ be such that $[D, [[\ldots[D, a_1], \ldots], a_{n-1}]] \in \mathcal{D}_{n-1}$, for all $a_1, \ldots, a_{n-1} \in \Omega^1(M)$, and $\mathrm{Simb}_n([D, d]) = 0$. Then, the induced bracket on functions $\{., \ldots, .\}_D$, is a Nambu–Poisson bracket.*

Now, we prove that any Nambu–Poisson bracket comes from an operator of this kind.

Theorem 13. *Let $\{., \ldots, .\}$ be a Nambu–Poisson bracket on $C^\infty(M) \times \overset{n}{\cdots} \times C^\infty(M)$. Then, it coincides with $\{., \ldots, .\}_D$, where $D = \mathcal{L}_P$ and P is the Nambu–Poisson n-multivector associated with $\{., \ldots, .\}$.*

Proof. If $f_1, \ldots, f_n \in C^\infty(M)$, we have

$$\begin{aligned}\{f_1, \ldots, f_n\}_{\mathcal{L}_P} &= [\![df_1, \ldots, df_{n-1}, f_n]\!]_{\mathcal{L}_P} \\ &= [[[\ldots[\mathcal{L}_P, df_1], \ldots], df_{n-1}], f_n](1) = \mathcal{L}_{i_{df_{n-1}\wedge\cdots\wedge df_1}P} f_n \\ &= (i_{df_{n-1}\wedge\cdots\wedge df_1}P)(df_n) = i_{df_n}(i_{df_{n-1}\wedge\cdots\wedge df_1}P) = P(df_1 \wedge \cdots \wedge df_{n-1} \wedge df_n) \\ &= \{f_1, \ldots, f_n\}.\end{aligned}$$

□

Our last result shows that, in a sense, the operators of the form $D = \mathcal{L}_P$ are the unique ones for which $\{., \ldots, .\}_D$ is a Nambu–Poisson structure. We will need some technical results first.

Lemma 14 (of localization). *Let $p : E \longrightarrow M$ be a vector bundle with finite rank over a manifold M. Let $\mathcal{E}$ be the space of its sections. Then, given a linear map $A : \mathcal{E} \longrightarrow C^\infty(M)$, there exists a (necessarily unique) section α of the dual vector bundle $E^* \longrightarrow M$ such that, for any point $x \in M$ and any element X in $\mathcal{E}$,*

$$A(X)(x) = \alpha(X(x))$$

if and only if A is $C^\infty(M)$-linear.

(This is a standard result in differential geometry, see e.g. [War 71, pp. 64–5]).

Definition 15. *A differential operator D (of any order and degree) is said to be tensorial if, for all $f \in C^\infty(M)$,*

$$[D, f] = 0.$$

Lemma 16. *Let $D \in \mathcal{D}_n^{-(n-1)}$ be a tensorial operator. Then, there exist unique $A \in \Gamma(\Lambda^{n-1}TM))$, $\Delta \in \Gamma(\Lambda^n TM \otimes T^*M)$ such that*

$$D = i_A + i_\Delta.$$

Proof. As a consequence of $[D, f] = 0$, we have that for all $\alpha \in \Omega^{n-1}(M)$, $D(f\alpha) = fD(\alpha)$, so $D|_{\Omega^{n-1}(M)} : \Omega^{n-1}(M) \longrightarrow C^\infty(M)$ is $C^\infty(M)$-linear and—by the localization lemma—it defines an $A \in \Gamma(\Lambda^{n-1}TM))$ such that

$$D|_{\Omega^{n-1}(M)} = i_A|_{\Omega^{n-1}(M)}.$$

Next, we study what happens when $D - i_A$ acts on n-forms. Just because D and i_A are so, $D - i_A$ is a tensorial operator, and then

$$\begin{aligned} \tilde{\Delta} : \Omega^1(M) \times \overset{n)}{\cdots} \times \Omega^1(M) \times \mathcal{X}(M) &\longrightarrow C^\infty(M) \\ (a_1, \ldots, a_n, X) &\longmapsto ((D - i_A)(a_1 \wedge \cdots \wedge a_n))(X) \end{aligned}$$

is $C^\infty(M)$-linear in all its arguments. Then, again by the localization lemma, $\exists \Delta \in \Gamma(\Lambda^n TM \otimes T^*M)$ such that

$$i_\Delta|_{\Omega^n(M)} = D - i_A|_{\Omega^n(M)}.$$

Now, acting on any k-form, with $k < n - 1$, any of the previous operators gives 0 (note the degrees); thus, we have $D = i_A + i_\Delta$. □

Proposition 17. *Given $D \in \mathcal{D}_n^{-(n-1)}$, then* $\mathrm{Symb}_n([D, d]) = 0$ *if and only if* $\mathrm{Symb}_n(D) = \mathrm{Symb}(\mathcal{L}_N)$ *for some* $N \in \Gamma(\Lambda^n TM)$.

Proof. We will follow the ideas presented in [Bel-Mon 94], where the authors consider the $n = 2$ case.

If $\mathrm{Symb}_n(D) = \mathrm{Symb}(\mathcal{L}_N)$, where $N \in \Gamma(\Lambda^n TM)$, we can write $D = \mathcal{L}_N + \tilde{D}$, with $\tilde{D} \in \mathcal{D}_{n-1}$, and then, as $\mathcal{L}_N$ commutes with d,

$$[D, d] = [\tilde{D}, d] \in \mathcal{D}_{n-1}$$

so $\mathrm{Symb}_n([D, d]) = 0$.

For the converse, consider an arbitrary $D \in \mathcal{D}_n^{-(n-1)}$, neither necessarily verifying $\mathrm{Symb}_n([D, d]) = 0$, nor being tensorial. Let us determine its non-tensorial part by taking brackets with a function $f \in C^\infty(M)$. We have $[D, f] \in \mathcal{D}_{n-1}^{-(n-1)}$, so $\exists H_f \in \Gamma\Lambda^{n-1}(TM)$ such that $[D, f] = i_{H_f}$. Now, the mapping

$$\begin{aligned} H : C^\infty(M) &\to \Gamma\Lambda^{n-1}(TM) \\ f &\longmapsto H_f \end{aligned}$$

is a derivation: if $f, g \in C^\infty(M)$, by the Leibniz property for the bracket $[.,.]$,

$$i_{H_{fg}} = [D, fg] = f[D, g] + g[D, f] = f i_{H_g} + g i_{H_f}.$$

Now, to each derivation from $C^\infty(M)$ to $\Gamma\Lambda^{n-1}(TM)$, we can assign a $Q \in \Gamma(TM \otimes \Lambda^{n-1}(TM))$ in the following manner. Let

$$\begin{aligned} \tilde{Q} : \Gamma\Lambda^{n-1}(T^*M) &\to \Gamma(TM) \\ (a_1, \ldots, a_{n-1}) &\longmapsto \tilde{Q}(a_1, \ldots, a_{n-1}) : C^\infty(M) \to C^\infty(M) \\ f &\longmapsto i_{H_f}(a_1 \wedge \cdots \wedge a_{n-1}) = [D, f](a_1 \wedge \cdots \wedge a_{n-1}) \end{aligned}$$

and note that $\tilde{Q}(a_1, \ldots, a_{n-1}) \in \Gamma(TM)$ as a consequence of H being a derivation. Also, we have $i_{H_f}(ga_1 \wedge \cdots \wedge a_{n-1}) = g i_{H_f}(a_1 \wedge \cdots \wedge a_{n-1})$, so $\tilde{Q}(a_1, \ldots, a_{n-1})$ is $C^\infty(M)$-linear and there exists a $Q \in \Gamma(TM \otimes \Lambda^{n-1}(TM))$ such that

$$Q(df, a_1, \ldots, a_{n-1}) = i_{\tilde{Q}(a_1, \ldots, a_{n-1})} df = [D, f](a_1 \wedge \cdots \wedge a_{n-1}).$$

Let us study under what conditions this Q is skew symmetric. It suffices to only consider the case when $a_i = df_i \in \Omega^1(M)$, $i \in \{2, \ldots, n\}$, $f = f_1 \in C^\infty(M)$, and to observe that

$$Q(df_1, df_2, \ldots, df_n) = [[\ldots[[D, f_1], df_2], \ldots], df_n](1) = \Phi_D^n(f_1, df_2, \ldots, df_n).$$

Now, from the proof of proposition 10, we know that this is skew symmetric if and only if $\mathrm{Symb}_n([D, d]) = 0$. So, under this condition we can take

$$N = Q \in \Gamma(\Lambda^n(TM)).$$

In the last step, we check that, for any $f \in C^\infty(M)$, $[D - \mathcal{L}_N, f]$ is a tensorial operator. As $[D - \mathcal{L}_N, f] \in \mathcal{D}_{n-1}^{-(n-1)}$ and a differential operator is characterized by its action on forms of degree less or equal to its order, we only need to consider the case of an $(n-1)$-form. So, for $g_i \in C^\infty(M)$, $i \in \{1, n-1\}$, we compute

$$\begin{aligned}
&[D - \mathcal{L}_N, f](dg_1 \wedge \cdots \wedge dg_{n-1})\\
&\qquad = [D, f](dg_1 \wedge \cdots \wedge dg_{n-1}) - [\mathcal{L}_N, f](dg_1 \wedge \cdots \wedge dg_{n-1})\\
&\qquad = Q(df, dg_1, \ldots, dg_{n-1}) - i_{i_{df}N}(dg_1 \wedge \cdots \wedge dg_{n-1})\\
&\qquad = Q(df, dg_1, \ldots, dg_{n-1}) - N(df, dg_1, \ldots, dg_{n-1}) = 0.
\end{aligned}$$

Thus, $D - \mathcal{L}_N$ is a tensorial operator, and applying the previous lemma, $\exists A \in \Gamma(\Lambda^{n-1}TM))$, $\Delta \in \Gamma(\Lambda^n TM \otimes T^*M)$ such that

$$D = \mathcal{L}_N + i_A + i_\Delta$$

and then $[D, d] = \mathcal{L}_A + \mathcal{L}_\Delta$, with $\mathcal{L}_A \in \mathcal{D}_{n-1}^{-(n-2)}$ and $\mathcal{L}_\Delta \in \mathcal{D}_n^{-(n-2)}$. If $\mathrm{Symb}_n([D, d]) = 0$, then also $\mathcal{L}_\Delta = 0$, so $\Delta = 0$ and $\mathrm{Symb}_n(D) = \mathrm{Symb}(\mathcal{L}_N)$. □

Remark 18. For $n = 2$, this result appeared in [Bel 95].

Corollary 19. *Given $D \in \mathcal{D}_n^{-(n-1)}$, then $[D, d] = 0$ if and only if $D = \mathcal{L}_N$ for some $N \in \Gamma(\Lambda^n TM)$.*

Proof. If $D = \mathcal{L}_N$ for some $N \in \Gamma(\Lambda^n TM)$, it is clear that $[D, d] = 0$. For the converse, we have in the proof of the preceding proposition $0 = [D, d] = \mathcal{L}_A + \mathcal{L}_\Delta$, with $\mathcal{L}_A \in \mathcal{D}_{n-1}^{-(n-2)}$ and $\mathcal{L}_\Delta \in \mathcal{D}_n^{-(n-2)}$, so $\Delta = 0$, $A = 0$ and $D = \mathcal{L}_N$. □

7. Examples

We present here two examples of Nambu–Poisson structures with their corresponding n-Lie algebroids. Since it is unusual to find explicit examples in the literature, we shall be rather detailed.

7.0.1. Kepler dynamics. This example is based on [MVV 98]. It is well known that the Kepler dynamics has five first integrals, which are given by the components of the angular momentum and those of the Runge–Lenz vector. In action-angle coordinates J, φ, such integrals are $J_1, J_2, J_3, \varphi_1 - \varphi_2, \varphi_2 - \varphi_3$. Call them h_1, h_2, h_3, h_4, h_5 respectively. In the space M with coordinates J, φ, consider the Nambu–Poisson 6-vector

$$P = \frac{2mk^2}{(J_1 + J_2 + J_3)^3} \frac{\partial}{\partial J_1} \wedge \frac{\partial}{\partial J_2} \wedge \frac{\partial}{\partial J_3} \wedge \frac{\partial}{\partial \varphi_1} \wedge \frac{\partial}{\partial \varphi_2} \wedge \frac{\partial}{\partial \varphi_3} \tag{9}$$

and the induced Nambu–Poisson bracket

$$\{f_1, f_2, f_3, f_4, f_5, f_6\} = P(df_1 \wedge df_2 \wedge df_3 \wedge df_4 \wedge df_5 \wedge df_6).$$

It is immediate to prove that the Hamiltonian vector field corresponding to the multi-Hamiltonian $(h_1, h_2, h_3, h_4, h_5)$ is

$$X_{h_1,h_2,h_3,h_4,h_5} = \frac{2mk^2}{(J_1 + J_2 + J_3)^3} \left(\frac{\partial}{\partial \varphi_1} + \frac{\partial}{\partial \varphi_2} + \frac{\partial}{\partial \varphi_3} \right)$$

so that $X_{h_1,h_2,h_3,h_4,h_5}(g) = \{h_1, h_2, h_3, h_4, h_5, g\}$ for all $g \in C^\infty(M)$. Now, we can describe the associated Filippov algebroid. The vector bundle is $p : \Lambda^1(T^*M) \longrightarrow M$, the Filippov bracket on the space of sections $\Omega^1(M)$ is given by $\mathcal{L}_P$, where P is the Nambu–Poisson 6-vector (9); explicitly, we would write for $df_1, df_2, df_3, df_4, df_5, df_6 \in \Omega^1(M)$

$$
\begin{aligned}
&[\![df_1, df_2, df_3, df_4, df_5, df_6]\!]_{\mathcal{L}_P}\\
&\qquad = [[[[[[\mathcal{L}_P, df_1], df_2], df_3], df_4], df_5], df_6](1)\\
&\qquad = \mathcal{L}_{i_{df_1\wedge df_2\wedge df_3\wedge df_4\wedge df_5}P}(df_6)\\
&\qquad = d(P(df_1 \wedge df_2 \wedge df_3 \wedge df_4 \wedge df_5 \wedge df_6))\\
&\qquad = d\{f_1, f_2, f_3, f_4, f_5, f_6\}.
\end{aligned}
$$

Finally, the anchor map $q : \Omega^5(M) \longrightarrow \Gamma(TM)$, acts as

$$q(df_1 \wedge df_2 \wedge df_3 \wedge df_4 \wedge df_5) = [\![df_1, df_2, df_3, df_4, df_5, .]\!]_{\mathcal{L}_P}$$

thus

$$q(df_1 \wedge df_2 \wedge df_3 \wedge df_4 \wedge df_5)(g) = \{f_1, f_2, f_3, f_4, f_5, g\}.$$

7.0.2. Bi-Hamiltonian systems. This example is adapted from [Mag-Mag 91], and it is a system of the Calogero–Moser type. Consider two particles on a line, interacting through a potential proportional to the squared power of their distance

$$V = \frac{1}{(x_2 - x_1)^2}.$$

The Newtonian equations of motion are readily derived:

$$
\begin{aligned}
\ddot{x}_1 &= -\frac{2}{(x_2 - x_1)^3}\\
\ddot{x}_2 &= \frac{2}{(x_2 - x_1)^3}.
\end{aligned}
\tag{10}
$$

These equations can also be obtained from a Hamiltonian description. It suffices to take the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{(x_2 - x_1)^2}$$

and the canonical symplectic form on the phase space $T^*\mathbb{R}^2$, with coordinates (x_1, x_2, p_1, p_2):

$$\Omega = dx_1 \wedge dp_1 + dx_2 \wedge dp_2.$$

In fact, this system is a bi-Hamiltonian one. We could consider, along with H, another conserved quantity: the total momentum

$$K = p_1 + p_2$$

and the symplectic form

$$
\begin{aligned}
\Xi = (p_1 + \alpha)dx_1 \wedge dp_1 &+ (p_2 + \alpha)dx_2 \wedge dp_2\\
&+\beta dx_1 \wedge dx_2 + \alpha(dx_2 \wedge dp_1 + dx_1 \wedge dp_2) + \gamma dp_1 \wedge dp_2
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= \frac{p_1 - p_2}{4 + (x_1 - x_2)^2(p_1 - p_2)^2}\\
\beta &= \frac{2(x_1 - x_2)}{4 + (x_1 - x_2)^2(p_1 - p_2)^2}\\
\gamma &= \frac{2}{(x_1 - x_3)^3}
\end{aligned}
$$

and we would obtain the same evolution equations (10). What we are now going to do, is to construct a Nambu–Poisson system from this bi-Hamiltonian one. For this purpose, it is better to introduce new coordinates for the position of the particles:

$$z = x_1$$
$$r = x_2 - x_1$$

It is straightforward to prove that the Hamiltonians now adopt the form

$$H = p_z^2 + p_r p_z + \frac{1}{2} p_r^2 + \frac{1}{r^2}$$
$$K = 2p_z + p_r$$

(the expressions for the symplectic forms are, of course, also changed; for example now we have

$$\Omega = 2dz \wedge dp_z + dz \wedge dp_r + dr \wedge dp_z + dr \wedge dp_r).$$

Consider the Nambu–Poisson multivector

$$P = \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial p_z} \wedge \frac{\partial}{\partial p_r}$$

associated with which we have a Nambu–Poisson bracket $\{.\,, \ldots, .\}$, so we can compute the Hamiltonian vector field with respect to the 2-Hamiltonian (H, K): this is the vector field

$$X_{H,K} = -p_r \frac{\partial}{\partial r} + \frac{2}{r^3} \frac{\partial}{\partial p_z} - \frac{4}{r^3} \frac{\partial}{\partial p_r}$$

as can be readily seen by evaluating $\{H, K, g\} = P(dH \wedge dK \wedge dg)$ for an arbitrary function g. Then, it is quite easy to show the Filippov algebroid corresponding to this structure. It is given by the vector bundle $p : \Lambda^1(T^*M) \longrightarrow M$, and as in the previous example, the bracket on the space of sections is determined by $\mathcal{L}_P$:

$$\begin{aligned} [\![df_1, df_2, df_3]\!]_{\mathcal{L}_P} &= [[[\mathcal{L}_P, df_1], df_2], df_3](1) \\ &= \mathcal{L}_{i_{df_1 \wedge df_2} P}(df_3) \\ &= d(P(df_1 \wedge df_2 \wedge df_3)) \\ &= d\{f_1, f_2, f_3\} \end{aligned}$$

and the anchor map $q : \Omega^2(M) \longrightarrow \Gamma(TM)$, acts as

$$q(df_1 \wedge df_2) = [\![df_1, df_2, .]\!]_{\mathcal{L}_P}$$

where

$$q(df_1 \wedge df_2)(g) = \{f_1, f_2, g\}.$$

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